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Remarks on quantum vortex theory on Riemann surfaces

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Abstract

Quantized point vortex theories on a compact Riemann surface of arbitrary genus (in the zero total vorticity case) are investigated. By taking meromorphic functions thereon as order parameters and resorting to the Weil–Kostant, Abel, Riemann and Riemann–Roch theorems, a natural phase space and Hamiltonian for the vortex–antivortex configurations is exhibited, leading to explicit vortex–antivortex coherent states wave functions via geometric quantization.

Furthermore, a relationship between point and smooth vorticities is established by means of Green functions associated to divisors on a Riemann surface and Poincaré duality, thereby yielding a natural regularization of the singular theory. © 1998 Elsevier Science B.V.

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1. Introduction

In this paper we deal with vortex theory on a compact Riemann surface M of arbitrary genus g. Indeed, the traditional physical case amounts at considering the two-sphere ($g = 0$), thought of, as usual, as a compactified plane. But the general case possesses physical significance as well, and this justifies our approach, besides relying on a fully developed abstract theory. Indeed, in experiments, general Riemann surfaces can be produced

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using vycor glasses, whereupon a film of superfluid ^{4}He is laid down and adsorbed (see e.g. [13]).

Our basic idea is to interpret the point vortex case (a collection of vortices and antivortices on a Riemann surface M of genus g with vanishing total vorticity) as giving a (degree zero) divisor on M. An order parameter is, roughly speaking, a semiclassical feature of the quantized theory, which is meant to account for the relevant local behaviour of the medium under consideration. In our present context, *we shall take a meromorphic function on the surface as order parameter*, whose zeros and poles, counted according to their multiplicities, will describe the vortex–antivortex assembly. This already incorporates the Feynman–Onsager quantization condition (see e.g. [15]).

The problem of describing all meromorphic functions on a Riemann surface is a classical one: the explicit answer, straightforward in the case of the sphere (the order parameter is a rational function), requires, for higher genus, joint use of Abel's theorem and of the Riemann factorization theorem (together with constraints provided by the Weierstrass gap theorem) and it can be expressed in terms of theta functions (Theorem 3.1, see [7,8,14]).

The question of the mutual location of vortex–antivortex pairs will be tackled by resorting to the Riemann–Roch theory: given a system of n vortices (corresponding to a holomorphic line bundle (unique up to (holomorphic) isomorphism $L \to M$), their antivortex counterparts, *considered as a whole*, span the finite-dimensional vector space $H^0(L)$ consisting of all global holomorphic sections of $L \to M$. We work with nonspecial divisors, this implying the *topological invariance* of the Riemann–Roch space $H^0(L)$, whose dimension is then given by the Riemann–Roch formula (see Section 3). This is crucial since the dynamical group of vortex theory is $s\mathrm{Diff}(M)$. Thus, in the *higher genus* case the mutual positions of vortex–antivortex pairs are *not* arbitrary since the conditions dictated by Abel's and Weierstrass' theorems are to be enforced. Then Riemann–Roch describes the actual range of possibilities.

Summing up, an admissible collection of vortices (D_1) and antivortices ($-D_2$) is described by a meromorphic function (order parameter) $f_{D_1-D_2}$, which can be looked upon as a point in $X := \mathbf{P}(H^0(L)) \times \mathbf{P}(H^0(L))$ (projectivization is in order since a meromorphic function and any nonzero multiple thereof describe the same assembly).

Then we find an expression for the Hamiltonian of the system solely in terms of the order parameter $f_{D_1-D_2}$, involving Riemann's theta function (Theorem 4.1). This leads to the following conclusion:

The manifold X, parametrizing all admissible order parameters, and which comes endowed with a natural Kähler structure, is the natural phase space of the theory.

This allows us to take the successive step consisting in quantizing the theory. But this is now straightforward. Indeed *on performing (holomorphic) geometric quantization on X* (see e.g. the recent monographs [2,21]), *we explicitly realize the wave functions of vortex–antivortex configurations as* coherent states *of a holomorphic line bundle* obtained via pullback of the hyperplane section bundle on an appropriate projective space through a Segre map (Theorem 5.1) (see e.g. [3,17,18]). These coherent state wave functions enjoy transparent semiclassical properties, in view of the general theory (see e.g. [17,18,20]). The classical counterpart of the coherent state wave function is the order parameter (i.e. the point in X) attached to it, and we have a clear-cut mathematical formulation of the semiclassical meaning of the latter.

Our method partially explains why the traditional approaches to vortex quantization are so difficult to pursue: the point is that they take place in the vortex ambient space, whereas our proposal is to work in an abstract Riemann–Roch space. The actual structure of the order parameters is however encoded in a still very complicated Hamiltonian.

In the $g = 1$ case, we provide a concrete illustration of the general ideas involved (see Appendix A), treating the special, but important case of vortex theory on a torus (elliptic curve). In particular we discuss some physical consequences stemming from the Weierstrass gap theorem and from the classical geometric construction of the *group law on the elliptic curve* (see e.g. [7]).

We also wish to point out a link between 2D-vortex quantization in the delta-like and the smooth vorticity case, relying on the same circle of ideas. This problem has been tackled by resorting to different techniques in the two cases (and in 3D as well, [5,6,15,16,22]). A natural regularization of the singular vorticity can be obtained by considering a representative of the Poincaré dual of its divisor. In view of the so-called localization principle [1], its support can be shrunk to an arbitrary neighbourhood of the divisor. We give a formula for a canonical regularized velocity field in terms of a regularized Green function related to the bona fide Green function associated to any divisor and a two-form on M (see e.g. [10]). In the present situation the two-form involved is the Kähler form. The regularized vorticity form can be taken as an integer-valued linear combination of bump two-forms centred in the points of the singular vorticity divisor with nonoverlapping supports. In the framework of geometric quantization, the regularized velocity field can be interpreted as a globally defined connection one-form (Abelian gauge field) whose curvature is the regularized vorticity two-form. Moreover, in analogy to [11] (where the Riemann–Roch theorem has been already used in a different context to the purpose of understanding Haldane's fractional statistics, see [9] as well) we show that the order parameter above, when viewed as the unique (up to a scalar) holomorphic section of the holomorphically trivial line bundle above, furnishes the ground state of (an analogue of) an anyon Hamiltonian, given by the so-called rough Laplacian, which involves the connection given by the regularized velocity field. This connection turns out to coincide with the canonical hermitian and holomorphic connection of the same line bundle.

The present paper is organized as follows. In Section 2 we review the basic machinery of smooth vortex theory adapted to the two-dimensional case, giving, in particular, a formula for the Hamiltonian in terms of a regular Green function, to be employed later on. In Section 3 we set up the algebro-geometric machinery, settling the order parameter problem. Next (Section 4) we compute the singular Hamiltonian (deprived of self-interaction terms) and, in Section 5, we identify our natural dynamical system and proceed to its holomorphic geometric quantization. In Section 6 we discuss the regularization problem and exhibit the link with anyon theory hinted to above. Finally, Section 7 is devoted to conclusions and outlook and is followed by Appendix A on vortex theory on elliptic curves.

2. Nonsingular 2D-vortex theories

It is well known, after Marsden–Weinstein [12], that the natural geometric portrait of vortex dynamics (for perfect, i.e. inviscid, incompressible fluids) interprets the Euler equation (in vorticity form) as describing motion in a coadjoint orbit of the dual of the space of divergenceless vector fields of a manifold M (equipped with a volume form), labelled by the vorticity field. This space is (in a suitable technical sense) the Lie algebra of $s\mathrm{Diff}(M)$, the group of volume preserving diffeomorphisms of M.

Here we consider a compact Riemann surface (M, h) of genus g, with h a (Kähler) metric, giving rise to a Kähler (hence symplectic) two-form κ, which we assume normalized, i.e. $\int_M \kappa = 1$. In 2D one can identify the vorticity field with a (necessarily closed) two-form ω, and the corresponding velocity field with a coclosed one-form v (the metric is involved in these identifications). Let us write $\omega = \tilde{\omega}\kappa$, with $\tilde{\omega}$ a function. All objects involved are assumed to be smooth.

More precisely, we have the following:

Proposition 2.1.

(i) *A solution to the system*

$$\mathrm{d}v = \omega, \qquad \delta v = 0 \tag{2.1}$$

exists if and only if the total vorticity, given by $\int_M \omega$, vanishes.

(ii) *Given* (i), *there exists a unique solution to the system minimizing the Hamiltonian H of the theory, given by*

$$H := \int_M v \wedge *v \tag{2.2}$$

(*with $*$ the Hodge star, that is, H is the L^2-norm of v squared*), *and is given by the formula*

$$v = \delta\beta = \delta(G_{\mathrm{reg}}\kappa), \tag{2.3}$$

where G_{reg} is a regular Green function (*see below*).

(iii) *Moreover, the Hamiltonian reads*

$$H = \int_M G_{\mathrm{reg}}\kappa \wedge *\omega = \int_M G_{\mathrm{reg}}\tilde{\omega}\kappa. \tag{2.4}$$

Proof. The proof is an easy application of the Hodge decomposition theorem and of the Fredholm alternative. Let us first observe that (i) yields indeed the relevant physical case. Orthogonality of forms is defined as usual via the metric. The Hodge decomposition theorem states that any k-form ξ can be written as an orthogonal sum $\xi = \xi_{\mathrm{h}} + d\alpha + \delta\beta$ (ξ_{h} denoting the harmonic part of ξ). This first entails that ω is orthogonal to the (unique, up to a scalar) harmonic two-form κ, and this yields (i). This is also clear by cohomological reasons. Notice that the solution v is determined up to a harmonic one-form. After reformulating the above equation in terms of a *Poisson equation*:

$$\Delta v = \delta\omega, \tag{2.5}$$

where $\Delta = \delta d + d\delta$ is the Hodge Laplacian (at the level of one-forms), we find

$$v = \Delta^{-1}\delta\omega. \tag{2.6}$$

Here Δ^{-1} denotes the Green operator inverting the Hodge Laplacian on the appropriate subspace of $\Lambda^1(M)$. This is well defined since $\delta\omega$ is orthogonal to the space of harmonic one-forms. Recall that by the Hodge theorem, one has $\mathcal{H}^1(M) = H^1(M)$, and $b_1 := \dim H^1(M) = 2g$ (de Rham cohomology).

Now, again in view of the Hodge decomposition theorem, we may choose v such that $v_{\mathrm{h}} = 0$, and this choice clearly minimizes H. Moreover, it is straightforward to check that (replacing ξ by v) α can be set equal to zero, and β can be taken as $G_{\mathrm{reg}}\kappa$, $\Delta(G_{\mathrm{reg}}\kappa) = \omega$, with G_{reg} the *(regular) Green function* given by the unique solution equation (2.6) orthogonal to κ, Hodge-starred. This gives (ii). Finally, it is clear that H is given by (2.4). □

Actually, working in terms of de Rham's *currents*, the above setting can be rendered meaningful for singular vorticity as well. We shall relate the two cases (smooth and singular, respectively) via Poincaré duality and Green function theory in Section 6.

3. Singular 2D-vortex theories: A Riemann–Roch approach

In order to lend motivation to our subsequent analysis, let us briefly review the elementary example given by considering point vortices on the complex plane $\mathbb{C}$. The (divergence-free and irrotational) velocity field V can be given in terms of a (holomorphic) *complex potential* F via the formula (obvious notation)

$$V = V_1 + \mathrm{i}V_2 = \overline{F'}, \tag{3.1}$$

When taking

$$F := \frac{\gamma}{2\pi i}\log f \tag{3.2}$$

with f a rational function on $\mathbb{C}$ (with $\gamma \in \mathbb{R}$) – which can be taken as an order parameter – one can describe an arbitrary assembly of (point-like) vortices and antivortices (corresponding to the zeros and poles of f, respectively, counted according to their multiplicity (order)).

We shall say that a zero of order k gives a $k\gamma$-vortex, whereas a pole of order k gives a $(-k\gamma)$-antivortex. Extending f to the Riemann sphere yields the constraint $Z - P = 0$ (obvious notation). Rational functions on the Riemann sphere are one and the same thing as meromorphic functions.

Thus vortex intensities are naturally quantized: they are integer multiples of γ. This is consistent with the Feynman–Onsager quantization condition, which is to be implemented in order to yield a semiclassical interpretation of the order parameter. We set henceforth $\gamma = 1$.

The velocity one-form v is then given by the formula

$$v = \mathrm{Re}\left(\frac{1}{2\pi \mathrm{i}}\, \mathrm{d}\log f\right) \tag{3.3}$$

All this makes sense on any compact Riemann surface M.

Now we set up and discuss our basic correspondence between algebro-geometric and physical objects. We refer to any treatise devoted to Riemann surface theory for full details, e.g. [7,14].

An order parameter will be a meromorphic function f on M. The complex potential F and the velocity one-form v retain their expressions (3.2) and (3.3), respectively. An assembly of n point-like (i.e. singular) vortices and n anti-vortices, counted according to their multiplicity (order), will be represented by a *degree zero* divisor D. Recall that a *divisor* D is a finite formal sum of points on M with integer coefficients $\sum_i k_i P_i$. Any meromorphic function f on M gives rise to a degree zero divisor denoted by (f). The converse is true under suitable conditions we recall below. A divisor D is called *effective* if $k_i \geq 0$ for any i. The *degree* $\deg D$ of the divisor D is given by $\sum_i k_i$. An effective divisor D_{e} will describe a collection of vortices only (vorticity divisor).

We assume from the outset $n > 2g - 2$ (so we are dealing with *nonspecial* divisors). Let $L_{D_{\mathrm{e}}} \to M$, or simply $L \to M$ be a holomorphic line bundle associated with D_e (holomorphic isomorphism classes of holomorphic line bundles coinciding with linear equivalence classes of divisors). One has $n = \deg L$ (degree of L). The (*finite dimensional*) vector space $H^0(L)$ consisting of all global holomorphic sections of $L \to M$ over M is canonically identified with the vector space consisting of all meromorphic functions such that the divisor $(f) + D_{\mathrm{e}}$ is effective (this is indeed Riemann's original approach). Equivalently, $H^0(L)$ parametrizes all effective divisors linearly equivalent to D_{e}. The degree zero divisor $D = (f) = [D_{\mathrm{e}} + (f)] - D_{\mathrm{e}}$ with $f \in H^0(L)$ describes a *physical* vortex–antivortex system pertaining to D_{e}. The dimension of $H^0(L)$ (denoted by $h^0(L)$) can be computed by the *Riemann–Roch formula*

$$h^0(L) = n + 1 - g \tag{3.4}$$

and is clearly a *topological* invariant, so it does not change upon acting on M via $s\mathrm{Diff}(M)$ (see also the remarks in Section 5). However, what is actually more relevant is the projective space $\mathbf{P}(H^0(L))$ having dimension

$$\dim \mathbf{P}(H^0(L)) =: r = n - g. \tag{3.5}$$

Projectivization is needed since the zeros and poles of f coincide with those of $c \cdot f$, $c \in \mathbb{C}^*$. We shall resume these considerations in Section 5. Further geometric insight can be gained by resorting to the notion of complete linear system. This will be illustrated in the specific example treated in Appendix A.

Now the important point is that *not all degree zero divisors give rise to an order parameter, i.e. correspond to the zeros and poles of a meromorphic function.* Moreover, the *Weierstrass gap theorem* implies some constraints on the actual formation of vortex–antivortex configurations having special features: the simplest and most general one is that the multiplicity of a *single* anti-vortex should exceed the genus of the surface.

On denoting by $J(M)$ the Jacobian of M – which is a complex g-dimensional principally polarized abelian variety – by $A : M \to J(M)$ and ϑ the Abel–Jacobi map and Riemann's theta function on $J(M)$, respectively [7,14], we summarize the content of Abel's theorem and of the Riemann factorization theorem (see [7,14], in particular) in:

Theorem 3.1 (Abel, Jacobi, Riemann, Weierstrass). *Let $D = \sum_i (P_i - Q_i)$ be a divisor on M.*

(i) $g = 0$. *There exists an order parameter f if and only if D has degree zero, and it is given by a rational function having zeros in the P_i's and poles in the Q_i's.*

(ii) $g > 0$. *There exists an order parameter if and only if D has degree zero and is mapped to $0 \in J(M)$ by the Abel–Jacobi map A. The explicit answer is provided, up to a scalar, by the formula (valid for generic ζ belonging to the so called* theta divisor):

$$f(x) = \exp\left(\int_{P_0}^{x} \xi\right) \cdot \prod_{k=1}^{n} \frac{\vartheta(A(x) - A(P_k) - \zeta)}{\vartheta(A(x) - A(Q_k) - \zeta)}. \tag{3.6}$$

(iii) *Generically, the multiplicity of a single antivortex should exceed the genus of the surface.*

Remark. We assume all points to be distinct for notational convenience, as in the references quoted above. The one-form ξ in (3.6) is a suitable holomorphic one-form (Abelian differential) (see [14] for details).

4. The singular Hamiltonian

Before working out the singular Hamiltonian, we recall the standard notation $d = \partial + \bar{\partial}$, $d^c = (1/4\pi \mathrm{i})(\partial - \bar{\partial})$ and the Green function $G(\kappa, D)$ pertaining to a divisor D and the Kähler form κ, given by the distributional equation (see [10], we adopt a different sign convention):

$$dd^c G(\kappa, D) = \deg D\, \kappa + \delta_D \kappa, \qquad \int_M G(\kappa, D)\kappa = 0. \tag{4.1}$$

In our case $\deg D = 0$ so we are left with the singular vorticity $\delta_D\, \kappa$. Clearly if, in general, $D = \sum_i k_i P_i$, then $\delta_D = \sum_i k_i \delta_{P_i}$. We shall henceforth simply write $G(D)$ instead of $G(\kappa, D)$. Explicitly, in the case $D = P$, it reads

$$G(P) = \log |f|^2 + \alpha_P \tag{4.2}$$

with f a meromorphic function having a simple zero at P and defined in a *Zariski* open subset of M. The function f governs the singular behaviour at P whereas the function α_P

is smooth on M. This definition is readily extended by linearity to any divisor D. However, we ought to notice that for deg $D = 0$, only the singular part survives, hence we are left with the following (also observe that now $(f) = D$):

Proposition 4.1. *With the notation above, if $(f) = D$, we have*

$$G(D) = \log|f|^2 + constant, \tag{4.3}$$

which, after duly taking into account Theorem 3.1, can be expressed in terms of theta functions.

Splitting up again the Green function in terms of its point divisors ($D = \sum_i (P_i - Q_i)$), we may also write $G = G(D) = \sum_i G(P_i) - G(Q_i)$. One has, in general, the *reciprocity law*

$$G(P)(Q) = G(Q)(P) \tag{4.4}$$

for all $P \neq Q$. So we end up with the following expressions for the singular Hamiltonian H_s (passing to the distributional limit in (2.4), and after discarding the "self-interaction terms"), whose physical meaning is transparent:

Theorem 4.1.
(i) *The Hamiltonian reads, in the singular case*

$$H_s = \delta_D(G(D)) = 2\sum_{i<j}[G(P_i)(P_j) + G(Q_i)(Q_j)] - 2\sum_{i\leq j} G(P_i)(Q_j). \tag{4.5}$$

(ii) *The above formula can be written in terms of theta:*

$$\begin{aligned} H_s = 2\Bigg\{ &\sum_{i\neq h} \log|\vartheta(A(P_h) - A(P_i) - \zeta)| - \sum_{i,h} \log|\vartheta(A(P_h) - A(Q_i) - \zeta)| \\ &- \sum_{i,k} \log|\vartheta(A(Q_k) - A(P_i) - \zeta)| + \sum_{i\neq k} \log|\vartheta(A(Q_k) \\ &- A(Q_i) - \zeta)| + \sum_k Re\int_{Q_k}^{P_k} \xi \Bigg\} \end{aligned} \tag{4.6}$$

5. The phase space for singular vortex dynamics and its geometric quantization

We now proceed to the construction of the phase space for singular vortex dynamics. Notations are those of Section 3. Fix (admissible) vorticity (effective) divisors D_e^0 (reference configuration), D_e^1, D_e^2. Set $D_e^i - D_e^0 = (f_{D_e^i - D_e^0})$. Then

$$f_{D_e^2-D_e^1} = \frac{f_{D_e^2-D_e^0}}{f_{D_e^1-D_e^0}}. \tag{5.1}$$

Clearly there is no dependence on D_e^0. Now fix a basis $\{f_i\}$ of $H^0(L)$. We take $L = L_{D_e^0}$. Then any $f \in H^0(L)$ has the form $f = \sum_{i=0}^{r} \lambda_i f_i$. If f is nontrivial, not all λ_i's are zero, so we get a point in the projective space $\mathbf{P}(H^0(L))$. We set $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r)$ and also, with a slight abuse of language, $\lambda = (\lambda_0 : \lambda_1 : \ldots : \lambda_r)$. Moreover, a dot $\cdot$ will denote the scalar product in $\mathbb{C}^{r+1}$, and $|\cdot|$ its induced norm. On varying D_e^1 and D_e^2 independently, we may associate $f_{D_e^2-D_e^1}$ with a point $(\lambda, \mu) \in X := \mathbf{P}(H^0(L)) \times \mathbf{P}(H^0(L))$. We saw in Section 4 that (singular) vortex Hamiltonian takes the form $H = \delta_D(G(D))$ (here D is the full vorticity divisor) and can thus be viewed as a function (with singularities) on X. The space X is equipped with a symplectic-Kähler form Ω given by the sum of the two Fubini-Study forms of each copy of $\mathbf{P}(H^0(L))$. The following definition is now natural:

Definition 5.1. The dynamical system describing the motion of an assembly of an equal number of vortices and antivortices on a Riemann surface is, with the above notations:

$$(X, \Omega, H_s). \tag{5.2}$$

Remarks.

(i) The manifold X is not a coadjoint orbit of $s\mathrm{Diff}(M)$. Moreover, another reason why we choose to work with this phase space is tied to our insistence on having a *complex structure* on M compatible with the symplectic (area) form, which would be the minimal ingredient. This complex structure is in turn related to the (Riemannian) metric of M. The latter is necessary from a physical viewpoint since our surface is embedded in three-space and hence inherits a natural metric which can be taken as a reference metric. However, the Hamiltonian flow just preserves oriented areas. Nonetheless, we should be able to write down an order parameter for the theory *in terms of the reference data at each time*, but this requires the global vortex–antivortex motion to be governed by Abel's theorem: every configuration D must fulfill $A(D) = 0$. Indeed, Theorem 4.1 shows that the Hamiltonian just depends on such D's, thereby giving rise to a function on X.

(ii) We again stress the fact that under the condition $n > 2g - 2$ the dimension $h^0(L)$ is a *topological* invariant, so it is preserved under area preserving diffeomorphisms of M. This is necessary for the present approach to be meaningful. Of course, such diffeomorphisms do not preserve the metric, in general. Equation $r = n - g$ shows transparently how the presence of handles in M affects the number of degrees of freedom of our system. This fact could, in principle, be tested experimentally in that both g and n, for a vortex gas confined on a vycor medium, can be macroscopically large. We remark that the special divisor case could be physically relevant as well: $h^0(L)$ would then not be a constant of motion, in principle.

Let us now perform geometric quantization of (X, Ω). This will be achieved by the following:

Theorem 5.1.

(i) *The geometric quantization bundle $\mathcal{L}$ pertaining to (X, Ω) is the box product $\mathcal{O}(1)\boxtimes\mathcal{O}(1)$ of two hyperplane section bundles (dual to the tautological bundle) $\mathcal{O}(1)$. The quantum Hilbert space is $H^0(\mathcal{L})$, with dimension $h^0(\mathcal{L}) = (r+1)^2$.*

(ii) *Natural (normalized) wave functions (holomorphic sections) describing the vortex–antivortex assembly are provided by the so-called* coherent states $\Psi_{(\lambda,\mu)}$ *which are explicitly given by*

$$\Psi_{(\lambda,\mu)}(\lambda',\mu') = \frac{\overline{\lambda}\cdot\lambda'}{|\lambda|}\cdot\frac{\overline{\mu}\cdot\mu'}{|\mu|}. \tag{5.3}$$

Moreover, the scalar product of two coherent state wave functions takes the form

$$\langle\Psi_{(\lambda,\mu)},\Psi_{(\lambda',\mu')}\rangle = \frac{\overline{\lambda}\cdot\lambda'}{|\lambda||\lambda'|}\cdot\frac{\overline{\mu}\cdot\mu'}{|\mu||\mu'|}. \tag{5.4}$$

Proof.

(i) The assertion follows easily from the observation that $\mathcal{L}$ is the pull-back of the hyperplane section bundle $\mathcal{O}(1)$ over $\mathbf{P}^{r(r+2)}$ under the *Segre map* $S : \mathbf{P}^r \times \mathbf{P}^r \to \mathbf{P}^{r(r+2)}$, given explicitly by $((z_i),(w_j)) \mapsto (z_i w_j)$ (homogeneous coordinates involved), after recalling that one has, in the present situation, $h^0(\mathcal{O}(1)) = h^0(L) = r+1$ (see e.g. [7], and cf. also [4]).

(ii) We just have to observe that the formulae yielding the coherent states (which are nothing but hyperplane sections) can be simply inferred from elementary algebraic geometry, or by resorting to the general expression yielding them in the holomorphic case via a (sesquiholomorphically extended) Kähler potential ϕ (see also [3,4,16–18]):

$$\langle z, w\rangle = \mathrm{e}^{\phi(z,w)-(1/2)\phi(z,z)-(1/2)\phi(w,w)}, \tag{5.5}$$

and applying it to our situation. Canonical choices of ϕ can be obtained via the so-called Calabi's diastasis function (see e.g. [3]). □

Remark. Observe that H_s still remains singular when vortices and/or antivortices come together, but in this case the semiclassical description is no longer adequate. However, one can devise a regularization procedure through a cut-off, and, moreover, in view of a theorem of Cahen et al. [3], one can choose a quantizable function arbitrarily close (in the sup-norm topology) to our regularized Hamiltonian, since the phase space is compact Kähler and the quantization regular [18]. This can be achieved by constructing a regularized Hamiltonian through a regularized Green function related to a smooth vorticity ω_D representing the Poincaré dual to D (see Section 6). Recall that a function is quantizable in the framework of holomorphic geometric quantization if its symplectic gradient preserves the anti-holomorphic polarization: this entails that the quantum operator yielded by the standard prescription preserves holomorphic sections, that is, the quantum Hilbert space.

Finally observe that in this general framework one recovers the well-known property that the mean expectation value of a quantum observable in a coherent state is equal to its classical counterpart (cf. [3,20]. So this is true in particular for our approximate Hamiltonians.

6. Regularization of the singular theory

As far as the regularization problem is concerned, let us first consider the following example: let $\omega = \exp(-\rho^2/2)\mathbf{e}_3$ be a Gaussian-like regularization of a delta-like vorticity in the origin of $\mathbb{C}$ (with an auxiliary extra dimension adjoined). A standard calculation of its divergence-free velocity field (which will vanish both at the origin and at infinity) using polar coordinates yields $v_\rho = 0$ (radial component) and $v_\phi = \rho^{-1}(1 - \exp(-\rho^2/2))$ (angular component).

We generalize this example in the present context as follows. One can easily devise an analogous set-up for the regularized theory, again by using the Hodge theorem. Again D must have degree zero. We shall denote a representative of the Poincaré dual of the divisor, which we take as a regularized vorticity, by ω_D. For instance, one can take a bump two-form with support in an arbitrary neighbourhood of D.

One abuts at a regularized Green function $G_{\text{reg}}(D)$ fulfilling

$$\Delta G_{\text{reg}}(D) = \omega_D, \qquad \int_M G_{\text{reg}}(D)\, \kappa = 0. \tag{6.1}$$

A natural regularized velocity one-form is given by $v_{\text{reg}} = d^c G_{\text{reg}}$. In fact, one has by definition $\mathrm{d}v_{\text{reg}} = \omega_D$ and $\delta v_{reg} = 0$, since it is readily checked that $\delta d^c = 0$. This is of course consistent with our previous treatment of the nonsingular theory, Section 2.

At the geometric quantization level, the unique (up to a scalar) order parameter in $H^0(L_D)$ appears as the ground state of the so-called *rough* Laplacian $\Delta_{\mathrm{R}} = \overline{\nabla}^*\overline{\nabla}$ (differing from the complete, or Laplace–Beltrami–Hodge Laplacian by a curvature term given by the vorticity (this is a special instance of a *Weitzenböck formula*); this is to be compared with the anyon *quantum* Hamiltonians discussed in e.g. [11]). Here $\overline{\nabla}$ denotes the antiholomorphic part of the Chern–Bott connection (which coincides with the $\bar\partial$ operator in a holomorphic frame) attached to the line bundle determined by the vorticity form ω through the Weil–Kostant theorem (see e.g. [7,21]). We summarize the preceding discussion in the following

Theorem 6.1. *Let the degree zero vorticity divisor D fulfill the conditions dictated by Weiertrass' and Abel's theorems. Then*

(i) *There always exists a regularized vorticity field, Poincaré dual to the vorticity divisor. Its support can be shrunk to any neighbourhood of the divisor.*

(ii) *There exists a global canonical velocity field yielding the regularized vorticity minimizing the Hamiltonian H given by the following expression:*

$$v = d^c G_{\text{reg}}, \tag{6.2}$$

where G_{reg} is the regularized Green function associated to the bump vorticity two-form regularizing the singular vorticity.

(iii) *In terms of geometric quantization, the regularized velocity field becomes the canonical hermitian and holomorphic connection (the Chern–Bott connection) defined on the topologically and holomorphically trivial line bundle defined by the divisor up to isomorphism.*

(iv) *The unique (up to a factor) order parameter corresponds to the ground state of the rough Laplacian.*

Remark. We explicitly point out that the geometric quantization scheme has been applied *three times* in this paper, but in different guises. First it was used to give the Riemann–Roch space $H^0(L)$ ($L = L_{D_e}$). Then it was applied again to the projective space $\mathbf{P}(H^0(L))$, or, rather, to X, giving rise to $\mathcal{O}(1)\boxtimes\mathcal{O}(1)$. Finally, we also noticed that the *one-dimensional* geometric quantization space $H^0(L_D)$ yields the ground state of an anyon-like *quantized* Hamiltonian having a vivid geometric interpretation, which however has only an indirect relationship with the vortex Hamiltonian.

7. Concluding remarks

In this paper we tried and cast some light on quantum vortex theory on a Riemann surface in a geometric fashion building on the Riemann–Roch theorem and related algebro-geometric techniques.

We exhibited a natural phase space describing vortex–antivortex configurations, admitting a natural quantization and naturally leading to coherent states wave functions. This adheres to, and supplements the treatment of [5,6,22]. The basic message is that vortices and antivortices should behave as a whole, and their actual locations are described by points in a suitable projective space (a product thereof) yielded by the Riemann–Roch theory. The latter theory also reveals some constraints in actual formation of vortices and antivortices possessing special features.

It is also likely that the analogy with anyons hinted at here can be pursued much farther using the techniques employed in this paper, particularly in view of a deeper understanding of Laughlin's anyon wave functions (see e.g. [11]). We finally observe that the consideration of the anyon-like Hamiltonian is close in spirit with the construction of the irreducible representations of the canonical commutation relations described in [19]. The two constructions should be related by the Abel–Jacobi map. These problems will be possibly tackled elsewhere.

Appendix A. An application: Vortices on an elliptic curve

We shall work out the case $n = 3$, $g = 1$ (so $r = 2$) in some detail (the case $n = 1$ is ruled out by the Weierstrass gap theorem): this is the standard plane Weierstrass cubic. We refer to [7] for notations.

Consider a torus (elliptic curve) $M = \mathcal{C}$ in $\mathbf{P}^2$. This is complex-analytically isomorphic with $J(M)$, but should not be confused with it in the present discussion. A point on M is denoted by p, whereas we set $z = A(p)$ in $J(M)$, with the Abel–Jacobi map defined as in [7]. Let us check in this specific example the explicit dependence of the Hamiltonian on the projective parameters. The plane elliptic curve $M = \mathcal{C}$ is described by Weierstrass' equation

$$y^2 - 4x^3 + g_2x + g_3 = 0.$$

Contact with the general setup is established upon considering $D = 3p_0$, where $p_0 = (0 : 0 : 1)$ (so $A(p_0) = 0 \in J(M)$). Then the Riemann–Roch theorem yields immediately $h^0(L_D) = 3$.

So we have to look for meromorphic functions f such that $(f) + D \geq 0$. Then elliptic function theory yields $f = \lambda_0 1 + \lambda_1 \wp + \lambda_2 \wp'$. The construction of the cubic is an instance of the *Kodaira embedding theorem* applied to L_D. The coefficients $(\lambda_0, \lambda_1, \lambda_2) \neq (0, 0, 0)$ are homogeneous coordinates (Plücker coordinates) describing all lines in $\mathbf{P}^2$, which in turn furnish a complete linear system (of dimension 2).

Each line (with equation $\lambda_0 + \lambda_1 x + \lambda_2 y = 0$) intersects $\mathcal{C}$ in three points $\{p_i\}$ yielding, via the Abel–Jacobi map, three points $\{z_i\}$ in $J(M)$, and builds up an admissible vorticity divisor. On performing an analogous construction for the antivortices, we easily arrive at the required Hamiltonian (although the final formula looks cumbersome). Actually one has two equivalent expressions: in the first one, one writes $f = f_D$ in the form (obvious notation)

$$f = \frac{\lambda_0 + \lambda_1 \wp + \lambda_2 \wp'}{\mu_0 + \mu_1 \wp + \mu_2 \wp'}.$$

In the other one uses Riemann's theta function as in (3.6).

We now discuss some explicit examples concerning the physical interpretation of the *group law on the elliptic curve* (corresponding to the Lie group structure of the Jacobian). We recall that it implies the existence of precisely nine flexes on $\mathcal{C}$, and that any two flexes are collinear with a third one. Specifically, we have:

(i) It is not possible to have pairs consisting of a single vortex and antivortex.

(ii) A single antivortex has multiplicity ≥ 2. It can be provided by the Weierstrass $\wp$-function.

(iii) The group law on $\mathcal{C}$ can be physically realized upon considering a suitable meromorphic function f on $\mathcal{C}$, $f = \lambda_0 1 + \lambda_1 \wp + \lambda_2 \wp'$, having a triple pole at p_0 and three simple zeros at p_i, $i = 1, 2, 3$

(iv) Moreover, given a (-3)-antivortex on $\mathcal{C}$, there exist exactly eight $(+3)$-vortices matching it.

(v) It is possible to construct order parameters describing an assembly made up of one (-2)-antivortex located at p_0, corresponding to $z \in \Lambda$ and two $(+1)$-vortices whose images in $J(M)$ are located in symmetrical points with respect to either the centre of the fundamental parallelogram or the midpoints of its sides. These particular points (and only these) allow locations of $(+2)$-vortices (actually, on their corresponding points on $\mathcal{C}$). At the same time, these points correspond to the three simple zeros ω_1, ω_2, ω_3 of $\wp'$, whence this latter function describes an assembly of one (-3)-antivortex in 0 and three $(+1)$-vortices in ω_i (again, their counterparts in $\mathcal{C}$).

(vi) Other noteworthy configurations are provided by the (-3)-antivortex in 0 and three $(+1)$-vortices in the *collinear* (in $\mathbb{C}$, and *also in* $\mathcal{C}$) flexes (five configurations). The remaining three configurations of collinear flexes (including p_0) are already encompassed in the ones discussed above.

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